Radion effective potential in the Brane-World

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Abstract

We show that in brane-world scenarios with warped extra dimensions, the Casimir force due to bulk matter fields may be sufficient to stabilize the radion field ϕ. In particular, we calculate one loop effective potential for ϕ induced by bulk gravitons and other possible massless bulk fields in the Randall-Sundrum background. This potential has a local extremum, which can be a maximum or a minimum depending on the detailed bulk matter content. If the parameters of the background are chosen so that the hierarchy problem is solved geometrically, then the radion mass induced by Casimir corrections is hierarchycally smaller than the TeV. Hence, in this important case, we must invoke an alternative mechanism (classical or nonperturbative) which gives the radion a sizable mass, to make it compatible with observations.

I. INTRODUCTION

It has been suggested that theories with extra dimensions may provide a solution to the hierarchy problem [1,2]. The idea is to introduce a d-dimensional internal space of large physical volume $\mathcal{V}$, so that the the effective lower dimensional Planck mass $m_{pl} \sim \mathcal{V}^{1/2} M^{(d+2)/2}$ is much larger than $M \sim TeV$- the true fundamental scale of the theory. In the original scenarios, only gravity was allowed to propagate in the higher dimensional bulk, whereas all other matter fields were confined to live on a lower dimensional brane. Randall and Sundrum [2] (RS) introduced a particularly attractive model where the gravitational field created by the branes is taken into account. Their background solution consists of two parallel flat branes, one with positive tension and another with negative tension embedded in a a five-dimensional Anti-de Sitter (AdS) bulk. In this model, the hierarchy problem is solved if the distance between branes is about 37 times the AdS radius and we live on the negative tension brane. More recently, scenarios where additional fields propagate in the bulk have been considered [3–5].

In principle, the distance between branes is a massless degree of freedom, the radion field ϕ. However, in order to make the theory compatible with observations this radion must be stabilized [6–10]. Clearly, all fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy, and it seems natural to investigate whether these could provide the stabilizing force which is needed. In this paper, we shall calculate the radion effective potential $V_{eff}(\phi)$ due to KK gravitons and other massless bulk fields. As we shall see, this effective potential has a rather non-trivial behaviour, which generically develops a local extremum. Depending on the detailed matter content, the extremum could be a maximum or a minimum, where the radion could sit. For the purposes of illustration, here we shall concentrate on the background geometry discussed by Randall and Sundrum, although our methods are also applicable to other geometries, such as the one introduced by Ovrut et al. in the context of eleven dimensional supergravity with one large extra dimension [11].

The paper is organized as follows. In Section II, we introduce the model and conventions. In Section III we calculate the effective potential due to conformally invariant bulk fields, and discuss the stabilization mechanism. Conformally invariant fields are convenient because of their simplicity, which allows a quick derivation of their contribution. Moreover, the backreaction of the Casimir energy on the geometry can be taken into consideration in this case. Section IV is devoted to the graviton contribution. For gravitons, conformal invariance is lost, and the evaluation of the effective potential is not straightforward. A method will be developed which relates the curved space effective potential to a suitable effective potential in flat space. The later takes the form of a sum over the contributions of an infinite tower of massive KK fields, whose mass spectrum $m_n(\phi)$ is a function of the brane separation (the discrete index n labels the infinite KK tower). Our conclusions are summarized in Section V.

Related calculations of the Casimir interaction amongst branes have been presented in an interesting paper by Fabinger and Horava [12]. In the concluding section we shall comment on the differences between their results and ours.

II. THE RANDALL-SUNDRUM MODEL AND THE RADION FIELD

To be definite, we shall focus attention on the brane-world model introduced by Randall and Sundrum [2]. In this model the metric in the bulk is anti-de Sitter space (AdS), whose (Euclidean) line element is given by

$$ds^2 = a^2(z)\eta_{ab}dx^a dx^b = a^2(z)\left[dz^2 + d\mathbf{x}^2\right] = dy^2 + a^2(z)d\mathbf{x}^2. \tag{2.1}$$

Here $a(z) = \ell/z$, where ℓ is the AdS radius. The branes are placed at arbitrary locations which we shall denote by z_+ and z_-, where the positive and negative signs refer to the positive and negative tension branes respectively ($z_+ < z_-$). The "canonically normalized" radion modulus ϕ - whose kinetic term contribution to the dimensionally reduced action on the positive tension brane is given by

$$\frac{1}{2}\int d^4x\sqrt{g_+}\,g_+^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi, \tag{2.2}$$

-is related to the proper distance $d = \Delta y$ between both branes in the following way [6]

$$\phi = (3M^3\ell/4\pi)^{1/2}e^{-d/\ell}.$$

Here, $M \sim TeV$ is the fundamental five-dimensional Planck mass. It is usually assumed that $\ell \sim M^{-1}$. Let us introduce the dimensionless radion

$$\lambda \equiv \left(\frac{4\pi}{3M^3\ell}\right)^{1/2}\phi = \frac{z_+}{z_-} = e^{-d/\ell},$$

which will also be refered to as *the hierarchy*. The effective four-dimensional Planck mass m_{pl} from the point of view of the negative tension brane is given by $m_{pl}^2 = M^3\ell(\lambda^{-2} - 1)$. With $d \sim 37\ell$, λ is the small number responsible for the discrepancy between m_{pl} and M.

At the classical level, the radion is massless. However, as we shall see, bulk fields give rise to a Casimir energy which depends on the interbrane separation. This induces an effective potential $V_{eff}(\phi)$ which by convention we take to be the energy density per unit physical volume on the positive tension brane, as a function of ϕ. This potential must be added to the kinetic term (2.2) in order to obtain the effective action for the radion:

$$S_{eff}[\phi] = \int d^4x\,a_+^4\left[\frac{1}{2}g_+^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V_{eff}(\lambda(\phi))\right]. \tag{2.3}$$

In the following Sections, we calculate the contributions to V_{eff} from conformally invariant bulk fields and from bulk gravitons.

III. RADION STABILIZATION BY MASSLESS BULK FIELDS

Let us start by considering the contribution to $V_{eff}(\phi)$ from conformally invariant massless bulk fields. Technically, this is much simpler than finding the contribution from bulk gravitons which will be discussed in the next section. Moreover the problem of backreaction of the Casimir energy onto the background can be taken into consideration in this case.

Consider, for instance, a conformally coupled scalar χ. This obeys the equation of motion

$$-\Box_g\chi + \frac{D-2}{4(D-1)}\,R\,\chi = 0. \tag{3.1}$$

where $\Box_g$ is the d'Alembertian operator in the metric (2.1) and R is the Ricci scalar. Here, we consider the case of arbitrary odd spacetime dimension D, with branes of co-dimension one. By changing the variable $\chi \to \hat{\chi} = a^{(D-2)/2}\chi$, the equation of motion for χ becomes

$$\Box\hat{\chi} = 0. \tag{3.2}$$

Here $\Box$ is the *flat space* d'Alembertian. It is customary to impose Z_2 symmetry on the bulk fields, which results in Neumann boundary conditions

$$\partial_z\hat{\chi} = 0,$$

at z_+ and z_-. The eigenvalues of the d'Alembertian subject to these conditions are given by

$$\lambda_{n,k}^2 = \left(\frac{n\pi}{L}\right)^2 + k^2, \tag{3.3}$$

where n is a positive integer, $L = z_- - z_+$ is the coordinate distance between both branes and k is the coordinate momentum parallel to the branes.

Similarly, we could consider the case of massless fermions in the RS background. The Dirac equation is conformally invariant [13], and the conformally rescaled components of the fermion obey the flat space equation (3.2) with Neumann boundary conditions. Thus, the spectrum (3.3) is also valid for massless fermions.

Let us now consider the Casimir energy density in the conformally related flat space problem. We shall first look at the effective potential per unit area on the brane, $\mathcal{A}$. For bosons, this is given

$$V_0^b = \frac{1}{2\mathcal{A}}\mathrm{Tr}\,\ln(-\Box/\mu^2). \tag{3.4}$$

Here μ is an arbitrary renormalization scale. Using zeta function regularization [See e.g. [14], or the discussion in Section IV for details], it is straightforward to show that

$$V_0^b(L) = \frac{(-1)^{\eta-1}}{(4\pi)^\eta\eta!}\left(\frac{\pi}{L}\right)^{D-1}\zeta_R'(1-D). \tag{3.5}$$

Here $\eta = (D-1)/2$, and ζ_R is the standard Riemann's zeta function. The contribution of a massless fermion is given by the same expression but with opposite sign:

$$V_0^f(L) = -V_0^b(L). \tag{3.6}$$

The expectation value of the energy momentum tensor is traceless in flat space for conformally invariant fields Moreover, because of the symmetries of our background, it must have the form [13]

$$\langle T^z_{\ z}\rangle_{flat} = (D-1)\rho_0(z), \quad \langle T^i_{\ j}\rangle_{flat} = -\rho_0(z)\,\delta^i_{\ j}.$$

By the conservation of energy-momentum, ρ_0 must be a constant, given by

$$\rho_0^{b,f} = \frac{V_0^{b,f}}{2L} = \mp\frac{A}{2L^D},$$

where the minus and plus signs refer to bosons and fermions respectively and we have introduced

$$A \equiv \frac{(-1)^\eta}{(4\pi)^\eta \eta!}\pi^{D-1}\zeta_R'(1-D) > 0.$$

Now, let us consider the curved space case. Since the bulk is of odd dimension, there is no anomaly [13] and the energy momentum tensor is traceless in the curved case too. This tensor is related to the flat space one by (see e.g. [13])

$$\langle T^\mu_{\ \nu}\rangle_g = a^{-D}\langle T^\mu_{\ \nu}\rangle_{flat}.$$

Hence, the energy density is given by

$$\rho = a^{-D}\rho_0. \tag{3.7}$$

The effective potential per unit physical volume on the positive tension brane is thus given by

$$V_{eff}(\lambda) = 2\, a_+^{1-D}\int a^D(z)\rho\, dz = \mp\ell^{1-D}\frac{A\lambda^{D-1}}{(1-\lambda)^{D-1}}. \tag{3.8}$$

Note that the background solution $a(z) = \ell/z$ has only been used in the very last step.

The previous expression for the effective potential takes into account the casimir energy of the bulk, but it is not complete because in general the effective potential receives additional contributions from both branes. As we shall discuss in more detail in Section IV, we can always add to V_{eff} terms which correspond to finite renormalization of the tension on both branes. These are proportional to λ^0 and λ^{D-1}. The coefficients in front of these two powers of λ cannot be determined from our calculation and can only be fixed by imposing suitable renormalization conditions which relate them to observables. Adding those terms and particularizing to the case of $D = 5$, we have

$$V_{eff}(\lambda) = \mp\ell^{-4}\left[\frac{A\lambda^4}{(1-\lambda)^4} + \alpha + \beta\lambda^4\right], \tag{3.9}$$

where $A \approx 2.46 \cdot 10^{-3}$. The values α and β can be obtained from the observed value of the "hierarchy", λ_{obs}, and the observed value of the effective four-dimensional cosmological constant, which we take to be zero. Thus, we take as our renormalization conditions

$$V_{eff}(\lambda_{obs}) = \frac{dV_{eff}}{d\lambda}(\lambda_{obs}) = 0. \tag{3.10}$$

If there are other bulk fields, such as the graviton, which give additional classical or quantum mechanical contributions to the radion potential, then those should be included in V_{eff}. From the renormalization conditions (3.10) the unknown coefficients α and β can be found, and
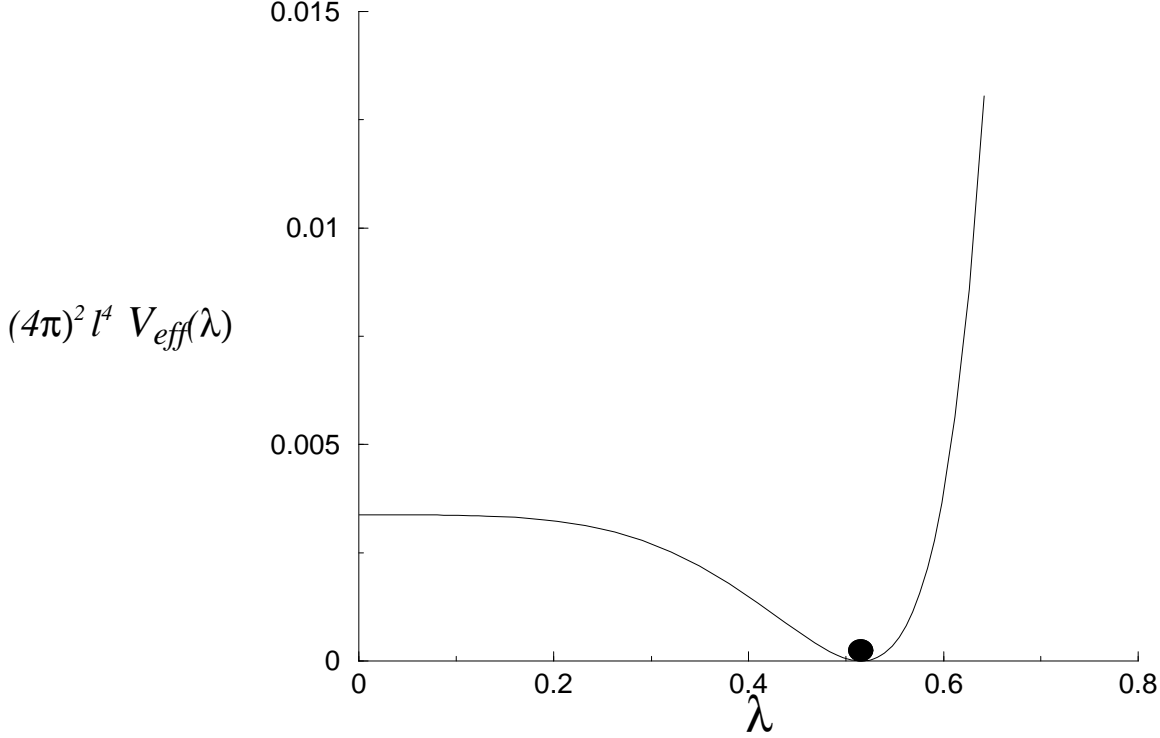
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FIG. 1. Contribution to the radion effective potential from a massless bulk fermion. This is plotted as a function of the dimensionless radion $\lambda = e^{-d/\ell}$, where d is the physical interbrane distance. The renormalization conditions (3.10) have been imposed in order to determine the coefficients α and β which appear in (3.9).

then the mass of the radion is calculable. In Fig. 1 we plot (3.9) for a fermionic field and a chosen value of λ_{obs}.

From (3.10), we have

$$\beta = -A(1 - \lambda_{obs})^{-5}, \quad \alpha = -\beta\lambda_{obs}^5. \tag{3.11}$$

These values correspond to changes $\delta\sigma_\pm$ on the positive and negative brane tensions which are related by the equation

$$\delta\sigma_+ = -\lambda_{obs}^5 \, \delta\sigma_-. \tag{3.12}$$

As we shall see below, Eq. (3.12) is just what is needed in order to have a static solution according to the five dimensional equations of motion, once the casimir energy is included.

We can now calculate the mass of the radion field $m_\phi^{(-)}$ from the point of view of the negative tension brane. For $\lambda_{obs} \ll 1$ we have:

$$m_\phi^{2\,(-)} = \lambda_{obs}^{-2} \, m_\phi^{2\,(+)} = \lambda_{obs}^{-2} \, \frac{d^2 V_{eff}}{d\phi^2} \approx \mp\lambda_{obs}\left(\frac{5\pi^3\zeta_R'(-4)}{6M^3 l^5}\right). \tag{3.13}$$

The contribution to the radion mass squared is negative for bosons and positive for fermions. Thus, depending on the matter content of the bulk, it is clear that the radion may be stabilized due to this effect.

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Note, however, that if the "observed" interbrane separation is large, then the induced mass is small. So if we try to solve the hierarchy problem geometrically with a large internal volume, then λ_{obs} is of order TeV/m_{pl} and the mass (3.13) is much smaller than the TeV scale. Such a light radion would seem to be in conflict with observations. In this case we must accept the existence of another stabilization mechanism (perhaps classical or nonperturbative) contributing a large mass to the radion. Of course, another possibility is to have λ_{obs} of order one, with M and ℓ of order m_{pl}, in which case the radion mass (3.13) would be very large, but then we must look for a different solution to the hierarchy problem.

Due to conformal invariance, it is straightforward to take into account the backreaction of the Casimir energy on the geometry. First of all, we note that the metric (2.1) is analogous to a Friedmann-Robertson-Walker metric, where the nontrivial direction is space-like instead of timelike. The dependence of a on the transverse direction can be found from the Friedmann equation

$$\left(\frac{a'}{a}\right)^2 = \frac{16\pi G_5}{3}\rho - \frac{\Lambda}{6}.$$
(3.14)

Here a prime indicates derivative with respect to the proper coordinate y [see eq. (2.1)], and $\Lambda < 0$ is the background cosmological constant. Combined with (3.7), which relates the energy density ρ to the scale factor a, Eq. (3.14) becomes a first order ordinary differential equation for a. We should also take into account the matching conditions at the boundaries

$$\left(\frac{a'}{a}\right)_{\pm} = \frac{\mp 8\pi G_5}{6}\sigma_{\pm}.$$
(3.15)

A static solution of Eqs. (3.14) and (3.15) can be found by a suitable adjustment of the brane tensions. Indeed, since the branes are flat, the value of the scale factor on the positive tension brane is conventional and we may take $a_+ = 1$. Now, the tension σ_+ can be chosen quite arbitrarily. Once this is done, Eq. (3.15) determines the derivative a'_+, and Eq. (3.14) determines the value of ρ_0. In turn, ρ_0 determines the co-moving interbrane distance L, and hence the location of the second brane. Finally, integrating (3.14) up to the second brane, the tension σ_- must be adjusted so that the matching condition (3.15) is satisfied. Thus, as with other stabilization scenarios, a single fine-tuning is needed in order to obtain a vanishing four-dimensional cosmological constant.

This is in fact the dynamics underlying our choice of renormalization conditions (3.10) which we used in order to determine α and β. Indeed, let us write $\sigma_+ = \sigma_0 + \delta\sigma_+$ and $\sigma_- = -\sigma_0 + \delta\sigma_-$, where $\sigma_0 = (3/4\pi\ell G_5)$ is the absolute value of the tension of the branes in the zeroth order background solution. Elliminating a'/a from (3.15) and (3.14), we easily recover the relation (3.12), which had previously been obtained by extremizing the effective potential and imposing zero effective four-dimensional cosmological constant (here, $\delta\sigma_{\pm}$ is treated as a small parameter, so that extremization of the effective action coincides with extremization of the effective potential on the background solution.) In that picture, the necessity of a single fine tuning is seen as follows. The tension on one of the walls can be chosen quite arbitrarily. For instance, we may freely pick a value for β, which renormalizes the tension of the brane located at z_-. Once this is given, the value of the interbrane distance

λ_{obs} is fixed by the first of Eqs. (3.11). Then, the value of α, which renormalizes the tension of the brane at z_+, must be fine-tuned to satisfy the second of Eqs. (3.10).

Eqs. (3.14) and (3.15) can of course be solved nonperturbatively. We may consider, for instance, the situation where there is no background cosmological constant ($\Lambda = 0$). In this case we easily obtain

$$a^3(z) = \frac{6\pi G A}{(z_- - z_+)^5}(C - z)^2 \tag{3.16}$$

where the brane tensions are given by $2\pi G \sigma_\pm = \pm(C - z_\pm)^{-1}$ and C is a constant. This is a self-consistent solution where the warp in the extra dimension is entirely due to the Casimir energy.

IV. GRAVITON CONTRIBUTION TO THE RADION EFFECTIVE POTENTIAL

Since the graviton is not conformally invariant, the techniques needed in order to find its contribution to the effective potential will have to be somewhat more sophisticated. However, because of the relevance of the graviton, it seems worthwile to pursue this task and in the following subsections we develop a convenient method to address this calculation. This consists of two steps. In Subsection IV.A we relate the curved space effective potential to a suitable flat space determinant. The calculation of this determinant is then presented in Section IV.B.

Our background has maximally symmetric foliations orthogonal to the y direction, and just like in the case of cosmological Friedmann-Robertson-Walker models, each graviton polarization contributes as a massless minimally coupled scalar field. Although this seems to be a well known fact which is often quoted in the literature, it is not always clear from the context what is the extent of this equivalence. The correspondence is straightforward at the classical level. At the quantum level, it can also be shown to hold, although this is not so straightforward to prove because careful gauge-fixing of the gravitational sector must be done. A detailed justification is lengthy, and will be presented elsewhere [15].

In short, the correspondence is as follows. Perturbations of the gravitational field in the Randall-Sundrum gauge can be expressed as

$$h_{ab} = g_{ab} - g_{ab}^{(0)} = \sum a^2 \sigma_{ab}^{(i)} \Phi_{(i)}, \tag{4.1}$$

where $\sigma_{55} = \sigma_{\mu\nu} = 0$ and $\sigma_{\mu\nu}$ is a constant transverse and traceless polarization tensor (a Fourier decomposition of Φ is assumed along the branes, in order to define transverse polarization). The summation is taken over all polarizations. The quadratic reduced action for one particular polarization becomes

$$W = \int d^D x \sqrt{g} \Phi \left(-\Box_g \Phi\right) + (\text{boundary term}), \tag{4.2}$$

where $\Box_g$ is the usual covariant scalar Laplacian associated with the five dimensional metric

$$g_{ab} = a^2 \eta_{ab},$$

and we omitted the index (i).

Thus, we shall be interested in the determinant of the operator

$$P = -\Box_g, \tag{4.3}$$

with appropriate boundary conditions at the branes. These are determined from Israel's junction conditions plus the requirement of Z_2 symmetry [16]. Equivalently, they can be found from the equation of motion for Φ in the RS background. It is easy to show that in terms of Φ, they reduce to the standard Neumann boundary conditions i.e.,

$$\partial_z \Phi = 0, \tag{4.4}$$

at $z = z_\pm$.

A. From flat to curved space

Our next task is to evaluate the determinant of the covariant d'Alembertian operator P defined in (4.3). Formally, this is the product of its eigenvalues. Nevertheless, in the present case, a straightforward calculation involving the eigenvalues of P seems rather complicated and not particularly illuminating. For instance, the eigenvalues of P are completely unrelated to the spectrum of Kaluza-Klein gravitons, and so the contribution from individual physical modes seems hard to identify.

A more promising strategy is to express $(\det P)$ in terms of the determinant of an operator whose eigenvalues are directly related to the KK masses. For this purpose, we introduce a one parameter family of metrics which interpolates between flat space and our physical space,

$$\tilde{g}_{ab}^\epsilon = \Omega_\epsilon^2 g_{ab}.$$

For definiteness we shall take

$$\Omega_\epsilon(z) = \frac{z}{\epsilon z + (1 - \epsilon)}, \tag{4.5}$$

where the parameter ϵ runs from 0 to 1. For $\epsilon = 0$ the fictitious metric $\tilde{g}_{ab}$ is flat space, whereas for $\epsilon = 1$ it coincides with the physical metric . The arguments presented in this section are independent of the precise form of the "path" (see Fig. 2). The explicit form (4.5) will only be used at the very end of the calculation, and it is chosen for convenience so that all metrics in the path are AdS spaces with curvature radius ℓ/ϵ.

Next, we introduce the operator P_ϵ defined by

$$\Omega_\epsilon^{\frac{D-2}{2}} P_\epsilon \Omega_\epsilon^{\frac{2-D}{2}} = \Omega_\epsilon^{-2} P = -\frac{1}{\Omega_\epsilon^2 a^2}\left(z^{D-2}\partial_z z^{2-D}\partial_z + \sum_{i=1}^{D-1}\partial_i^2\right). \tag{4.6}$$

This operator can be written in covariant form as

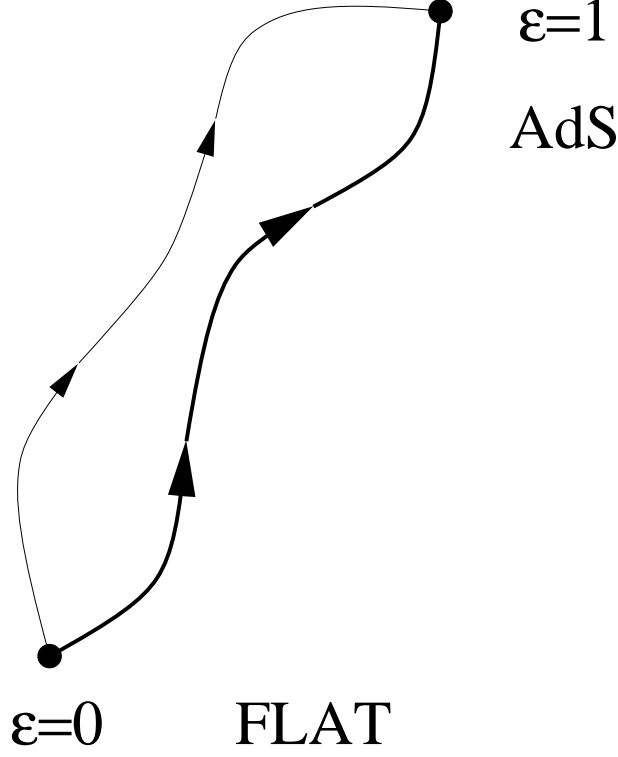$$P_\epsilon = -(\Box_{\tilde{g}} + E_\epsilon),$$

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FIG. 2. Paths from flat to AdS space in the space of conformal factors. The integral in (4.15) is independent on the path.

where $\square_{\tilde{g}}$ is the d'Alembertian in the space with metric $\tilde{g}_{ab}$ and

$$E_\epsilon = \frac{D-2}{4\Omega_\epsilon^2}\left[2\frac{\square_g\Omega_\epsilon}{\Omega_\epsilon} + (D-4)g^{ab}(\partial_a \ln\Omega_\epsilon)(\partial_b \ln\Omega_\epsilon)\right].$$

For later reference, we note that P_ϵ acts on the rescaled field

$$\tilde{\Phi} = \Omega_\epsilon^{\frac{2-D}{2}}\Phi, \tag{4.7}$$

which obeys the boundary condition $\mathcal{B}_\epsilon\tilde{\Phi} = (\tilde{n}^a\partial_a + S_\epsilon)\tilde{\Phi}|_{\partial M} = 0$. Here

$$S_\epsilon = \frac{D-2}{2}\tilde{n}^a\partial_a \ln\Omega_\epsilon, \tag{4.8}$$

and $\tilde{n}$ is the inward directed normal vector at the boundary.

We are interested in the effective potential per unit co-moving area $\mathcal{A}_c$. This is given by $V_{eff}^c = \mathcal{A}_c^{-1}\ln(\det P)^{1/2}$. As we shall see, the dependence of $(\det P_\epsilon)$ on ϵ along the conformal path in Fig. 2 is easily found in terms of local quantities. Hence, instead of dealing directly with $(\det P)$, we shall calculate the flat space determinant $(\det P_0)$, and then integrate the dependence on ϵ along the path. The effective potential per unit comoving area is given by

$$V_{eff}^c = V_{\epsilon=1},$$

where

$$V_\epsilon \equiv \frac{1}{2\mathcal{A}_c} \mathrm{Tr}\ln\left(\frac{P_\epsilon}{\mu^2}\right) = -\frac{1}{2\mathcal{A}_c}\lim_{\nu \to 0}\partial_\nu \zeta_\epsilon(\nu). \tag{4.9}$$

Here, we have introduced

$$\zeta_\epsilon(\nu) = \mathrm{Tr}\left(\frac{P_\epsilon}{\mu^2}\right)^{-\nu} = \frac{2\mu^{2\nu}}{\Gamma(\nu)}\int_0^\infty \frac{d\xi}{\xi}\xi^{2\nu}\mathrm{Tr}\left[e^{-\xi^2 P_\epsilon}\right]. \tag{4.10}$$

The symbol Tr refers to the usual L_2 trace. For any operator $\mathcal{O}$, the trace can be represented as

$$\mathrm{Tr}[\mathcal{O}] = \sum_i \int d^D x \, g^{1/2}\,\Phi_i(\mathcal{O}\Phi_i) = \sum_i \int d^D x \, \tilde{g}^{1/2}\,\tilde{\Phi}_i(\mathcal{O}\tilde{\Phi}_i),$$

where Φ_i (or $\tilde{\Phi}_i$) form an orthonormal basis with respect to the measure associated with the metric g (or $\tilde{g}$ respectively). It is straightforward to show that

$$\mathrm{Tr}\left[f e^{-\xi^2 P_\epsilon}\right] = \mathrm{Tr}\left[f e^{-\xi^2 \Omega_\epsilon^{-2} P}\right]. \tag{4.11}$$

Following the Ref. [17–20], we introduce the asymptotic expansion of the trace for small ξ:

$$\mathrm{Tr}\left[f(x)e^{-\xi^2 P_\epsilon}\right] \sim \sum_{n=0}^\infty \xi^{n-D} a_{n/2}(f, P_\epsilon), \tag{4.12}$$

where f is an arbitrary smooth function. This is a generalization of the more widely used De Witt-Schwinger expansion for the solution of the heat kernel equation [21](which corresponds to the case $f = 1$). The dependence of V_ϵ on ϵ is related to the coefficient $a_{D/2}$ in the following way

$$
\begin{aligned}
\partial_\epsilon \lim_{\nu \to 0}\partial_\nu \zeta_\epsilon(\nu) &= \lim_{\nu \to 0}\partial_\nu \frac{2\mu^{2\nu}}{\Gamma(\nu)}\int_0^\infty d\xi \, \xi^{2\nu}\mathrm{Tr}\left(-f_\epsilon \partial_\xi e^{-\xi^2 P_\epsilon}\right)\\
&= \lim_{\nu \to 0}\partial_\nu \frac{4\nu\mu^{2\nu}}{\Gamma(\nu)}\int_0^\infty d\xi \, \xi^{2\nu-D-1}\left[\xi^D \mathrm{Tr}\left(f_\epsilon e^{-\xi^2 P_\epsilon}\right)\right]\\
&\sim \lim_{\nu \to 0}\partial_\nu \frac{2\mu^{2\nu}\Gamma^{-1}(\nu)}{(D-2\nu)(D-1-2\nu)\cdots(1-2\nu)}\int_0^\infty d\xi \, \xi^{2\nu}\partial_\xi^{D+1}\sum_{n=0}^\infty \xi^n a_{n/2}(f_\epsilon, P_\epsilon)\\
&= 2a_{D/2}(f_\epsilon, P_\epsilon),
\end{aligned}
\tag{4.13}
$$

where we have defined

$$f_\epsilon = \partial_\epsilon \ln \Omega_\epsilon. \tag{4.14}$$

In the first equality, we have used (4.11). In the second equality, we performed integration by parts assuming $2\nu > D$. The contribution from $\xi = \infty$ vanishes due to the exponential suppression in $\mathrm{Tr}\left[f(x)e^{-\xi^2 P_\epsilon}\right]$. In the third equality, we performed a further integration by parts.

The dependence in ϵ can now be integrated along the path which joins flat space with AdS, and the effective potential can be expressed as

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$$V_{eff}^c = V_1 = V_0 - \frac{1}{\mathcal{A}_c} \int_0^1 d\epsilon \, a_{D/2}(f_\epsilon, P_\epsilon). \tag{4.15}$$

The calculation of V_0 is done in the next section. The coefficient $a_{D/2}$, for $D \leq 5$, has been studied in [17,20,22–25] for a large class of "covariant" operators which includes our P_ϵ. In general, this can be expressed in terms of integrals over the bulk and boundary of local invariants constructed from the metric $\tilde{g}$ and the functions S_ϵ and E_ϵ which enter the boundary condition, given in Eq. (4.8).

For the case of odd dimension D, the coefficients $a_{D/2}$ contain only contributions from the boundary. For instance, for $D = 3$ we have [22]

$$a_{3/2}(f, P_\epsilon) = \frac{1}{1536\pi} \int_{\partial\mathcal{M}} \{ f(96E_\epsilon + 16\tilde{R} - 8\tilde{R}^\mu{}_{n\mu n} + 13\tilde{K}^2 + 2\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu}$$
$$-96S_\epsilon\tilde{K} + 192S_\epsilon^2) + f_{;n}(-6\tilde{K} + 96\tilde{S}) + 24f_{;nn} \}.$$

Here $\tilde{K}_{\mu\nu}$ is the extrinsic curvature and $\tilde{R}^a_{bcd} = +\tilde{\Gamma}^a_{bd,c} - ...$ is the Riemann tensor. As usual, greek indices run from 0 to 3 and the index n indicates contraction with the inward unit normal vector $\tilde{n}$. If we integrate this expression along the path given by (4.15), then all intermediate geometries are AdS of radius ℓ/ϵ and the invariants are easily found. The result is given by

$$\frac{1}{\mathcal{A}_c} \int_0^1 d\epsilon \, a_{3/2}(f_\epsilon, P_\epsilon) = \frac{1}{4\pi z_+^2} \left[\frac{1}{16}(\ln z_+ + \lambda^2 \ln z_-) - \frac{3}{64}(1 + \lambda^2) \right], \tag{4.16}$$

where we have reintroduced the dimensionless radion $\lambda = z_+/z_-$. The general expression for $a_{5/2}$, relevant to the five dimensional case, has been given by Kirsten [26]. It is rather lengthy since it involves combinations of geometric invariants that contain 4 derivatives of the metric, and we shall not reproduce it here. Integrating this coefficient over ϵ, we obtain

$$\frac{1}{\mathcal{A}_c} \int_0^1 d\epsilon \, a_{5/2}(f_\epsilon, P_\epsilon) = \frac{1}{4\pi^2 z_+^4} \left[-\frac{27}{128}(\ln z_+ + \lambda^4 \ln z_-) + \frac{53}{5120}(1 + \lambda^4) \right]. \tag{4.17}$$

It should be noted that under the rescaling $z_\pm \to \vartheta z_\pm$ (with $\vartheta = const.$), which corresponds to a translation of both branes in the y direction, the expression for the effective potential V_{eff}^c must scale like

$$V_{eff}^c \to \vartheta^{1-D} V_{eff}^c. \tag{4.18}$$

This shift is a mere coordinate transformation which changes the physical volume corresponding to a given co-moving volume. We shall show in the next section that the quantity V_0 does not scale in this way, and that the integrals (4.16) and (4.17) have precisely the form which restores the scaling (4.18). For that reason, the explicit evaluation of these integrals is not strictly necessary, although of course it provides a valuable consistency check. As we shall see, the integrals can be guessed from the scaling property (4.18) up to terms which can be reabsorbed by finite renormalization.

B. Computation of $V_{eff}(\phi)$

From (4.6), the eigenvalue problem for $\epsilon = 0$ takes the form:

$$\ell^{-2}\left[z^{D-2}\partial_z z^{2-D}\partial_z - k^2\right]\Phi_i = -\lambda_i \Phi_i, \tag{4.19}$$

with boundary condition $\partial_z \Phi_i = 0$ at $z = z_\pm$. Here, an expansion in Fourier modes of wavenumber k along the flat branes has been assumed. In terms of the rescaled field $\tilde{\Phi}$ defined in (4.7), this equation looks like a Schrodinger equation with a "volcano"-type potential [16]. Defining $m_i^2 = -k^2 + \ell^2\lambda_i$, we have

$$\left[z^{D-2}\partial_z z^{2-D}\partial_z + m_i^2\right]\Phi_i = 0, \tag{4.20}$$

which has a general solution $\Phi_i = z^\eta \left(A_1 J_\eta(m_i z) + A_2 Y_\eta(m_i z)\right)$. The boundary conditions give an equation which determines m_i as

$$F(\tilde{m}_i) = J_{\eta-1}(\tilde{m}_i\lambda)Y_{\eta-1}(\tilde{m}_i) - Y_{\eta-1}(\tilde{m}_i\lambda)J_{\eta-1}(\tilde{m}_i) = 0, \tag{4.21}$$

where J and Y are Bessel functions, $\eta = (D-1)/2$, $\tilde{m}_i = m_i z_-$ and we have reintroduced $\lambda = z_+/z_-$.

Equation (4.21) gives the physical spectrum of masses $m_- = \tilde{m}_i/\ell$ for the KK gravitons from the point of view of the negative tension brane. This spectrum is a function of the interbrane distance, which enters Eq. (4.21) through the hierarchy λ. The zeta function for the operator P_0, given by

$$\zeta_0(\nu) = \mathcal{A}_c \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_i \left(\frac{k^2 + m_i^2}{\ell^2\mu^2}\right)^{-\nu}, \tag{4.22}$$

resembles the zeta function for a tower of fields of mass m_i in flat space. Nevertheless, this is to some extent just a formal analogy: instead of physical momentum and mass, the coordinate momenta k and "coordinate" mass eigenvalue m_i appear in this expression. [1]

Performing the momentum integrals in (4.22) we have

$$\zeta_0(\nu) = \mathcal{A}_c \frac{(\ell\mu)^{2\nu} z_-^{2\nu-(D-1)}\Gamma(\nu-\eta)}{(4\pi)^\eta\Gamma(\nu)}\hat{\zeta}(2\nu - (D-1)), \tag{4.23}$$

---

[1]In fact, it does not seem possible to write down an expression analogous to (4.22) for our V_{eff}, containing physical masses and momenta instead of coordinate ones. This would be possible if our space-time were ultrastatic (see e.g. [27]), which just means that the lapse function is a constant. However, in our case, time flows at different rates in different places, so for instance the physical mass depends on which brane we are considering. This seemingly trivial complication prevents us from doing the naive "projection" of the problem onto four dimensional slices, and renormalizing infinities as if we were considering an infinite tower of fields living in a flat space. In fact, such naive procedure would give a qualitatively different result for $V_{eff}(\phi)$ depending on whether we compute it on the positive or on the negative tension brane, which is clearly nonsensical. This is why the techniques of Section IV.A have been introduced.
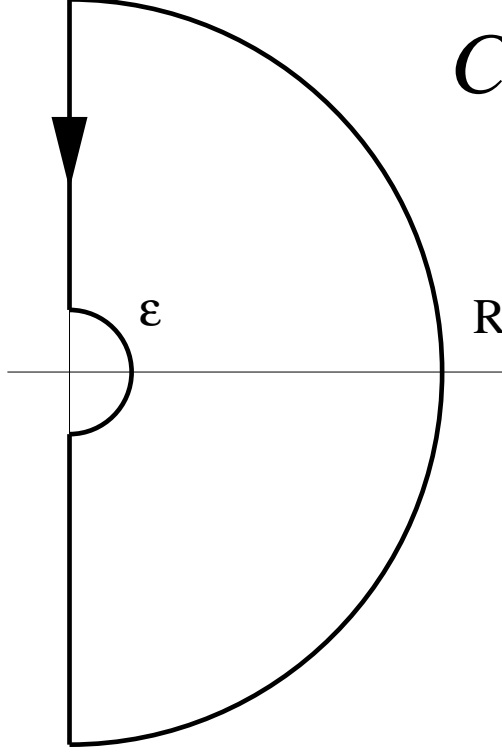
FIG. 3. Contour of integration in Eq. (4.26).

where we defined

$$\hat{\zeta}(s) = \sum_i \tilde{m}_i^{-s}. \tag{4.24}$$

Substituting (4.23) in (4.9) we have

$$V_0(\lambda) = \frac{(-1)^{\eta-1}}{(4\pi)^\eta \eta! z_-^{D-1}} \left[\left\{ \ln(\ell\mu z_-) + \frac{1}{2} \sum_{n=1}^{\eta} n^{-1} \right\} \hat{\zeta}(1-D) + \hat{\zeta}'(1-D) \right]. \tag{4.25}$$

Now the problem reduces to the calculation of the generalized zeta function (4.24). This is a discrete sum over the zeros of the function F defined in (4.21). This type of sums, where F is a combination of Bessel functions, have been previously studied in [28–31]. Following [29], we express $\hat{\zeta}(s)$ as

$$\hat{\zeta}(s) = \frac{1}{2\pi i} \int_{\mathcal{C}} dt\, t^{-s} \frac{F'}{F} = \frac{s}{2\pi i} \int_{\mathcal{C}} dt\, t^{-1-s} \ln F. \tag{4.26}$$

This follows from the fact that F has only simple zeros and these are on the positive real axis. The contour of integration $\mathcal{C}$ is given in Fig. 3. The expression (4.26) is convergent for sufficiently large s.

After some manipulations, which are deferred to Appendix A, we find

$$V_0(\lambda) = \frac{1}{(4\pi)^\eta (\eta-1)!\, z_+^{D-1}} \left[\mathcal{I}_K + \lambda^{D-1} \mathcal{I}_I + \beta_{D-1}(1+\lambda^{D-1}) \left(\ln(\ell\mu) + \frac{1}{2} \sum_{n=1}^{\eta-1} n^{-1} \right) \right.$$
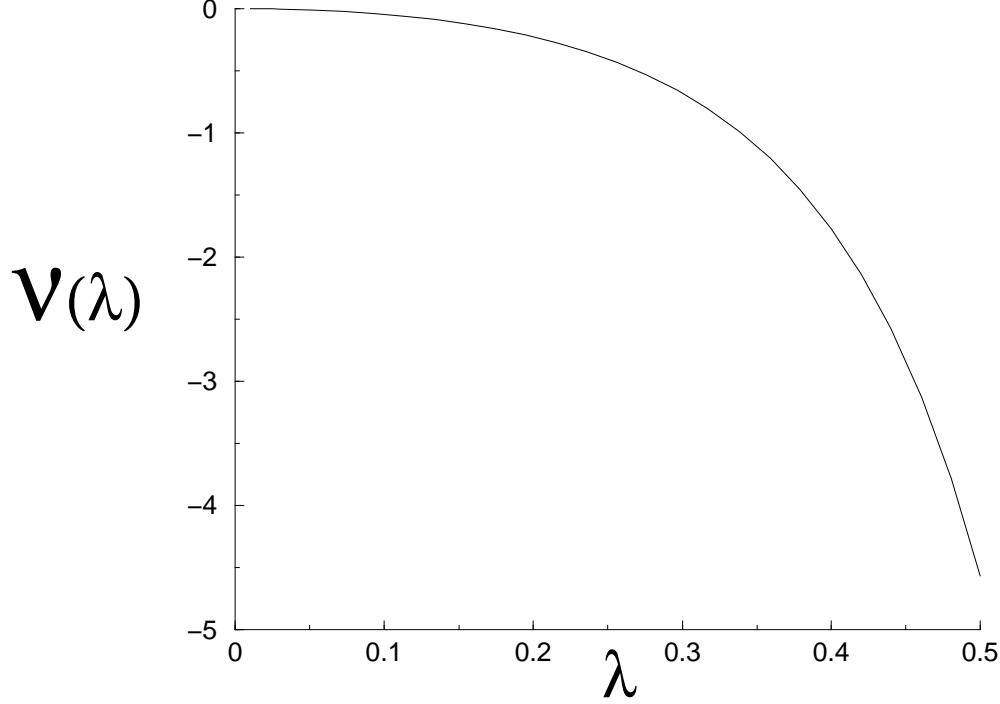
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FIG. 4. The function $\mathcal{V}(\lambda)$.

$$+\beta_{D-1}\left(\ln z_+ + \lambda^{D-1}\ln z_-\right) + \lambda^{D-1}\mathcal{V}(\lambda)\Big]. \tag{4.27}$$

Here $\mathcal{I}_I, \mathcal{I}_K$ and β_{D-1} are constants which are evaluated in the Appendix, and

$$\mathcal{V}(\lambda) = \int_0^\infty d\rho\, \rho^{D-2}\ln\left(1 - \frac{I_{\eta-1}(\lambda\rho)}{K_{\eta-1}(\lambda\rho)}\frac{K_{\eta-1}(\rho)}{I_{\eta-1}(\rho)}\right), \tag{4.28}$$

where I and K are the modified Bessel functions. The function $\mathcal{V}(\lambda)$ is plotted in Fig. 4.

The coefficients which accompany the powers λ^0 and λ^{D-1} will change if we change the renormalization scale μ inside the logarithm. This corresponds to the well known scaling of the effective potential under changes of μ, which adds a geometric term proportional to the coefficient $a_{D/2}(f=1,P)$ defined in (4.16). It is easy to show that this coefficient is Weyl invariant and therefore independent of ϵ. The invariant is made out of local terms (such as the square of the extrinsic curvature) which should also be included in the bare action. The renormalized couplings in front of these local terms are in fact supposed to be negligibly small (they have not been included in the equations of motion satisfied by the background), so the terms proportional to $\ln\mu$ might as well be dropped from V_0. On the other hand, we can always add to V_0 terms which correspond to a finite renormalization of the cosmological constant in the bulk. These will give contributions proportional to $(1-\lambda^{D-1})$. Also, one can add terms which correspond to a finite renormalizaton of the tension on the branes. These will give independent contributions proportional to λ^0 from the positive tension brane and to λ^{D-1} from the negative tension brane. Hence, the coefficients in front of these two powers

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of λ cannot be determined from the calculation and can only be fixed by imposing suitable renormalization conditions which relate them to observables.

By using (4.15) with the results (4.16) or (4.17) obtained in the preceding section, we can now evaluate V_{eff}^c. As mentioned at the end of the last subsection, the explicit calculation of the generalized Seeley-De Witt coefficients is not strictly necessary. It is known that $a_{D/2}$ for odd D has only contribution from the terms depending on the background quantities evaluated on the boundaries. Therefore, the shift in the effective potential due to the variation of ϵ must be given by the sum of a function of z_+ plus a function of z_-:

$$\frac{1}{\mathcal{A}_c} \int_0^1 d\epsilon\, a_{D/2}(f_\epsilon, P_\epsilon) = \mathcal{F}_+(z_+) + \mathcal{F}_-(z_-). \tag{4.29}$$

Imposing that V_{eff}^c has the correct scaling behaviour given in Eq. (4.18), the functional form $\mathcal{F}_\pm$ is determined up to the two coefficients in front of λ^0 and λ^{D-1} mentioned above, which have the same λ dependence as cosmological constants on the respective branes. Taking this into account, we have

$$V_{eff}^c = \frac{1}{(4\pi)^\eta(\eta-1)!z_+^{D-1}} \left[\alpha + \beta\lambda^{D-1} + \lambda^{D-1}\mathcal{V}(\lambda) \right]. \tag{4.30}$$

The "renormalized" values α and β can be obtained as before from the renormalization conditions (3.10). An example of an effective potential after these conditions have been imposed is shown in Fig. 5. Here

$$V_{eff}(\lambda) = a^{1-D}(z_+)V_{eff}^c(\lambda),$$

is the effective potential per unit physical volume on the positive tension brane, which appears in Eq. (2.3). (Note that, in fact, the conditions (3.10) do not depend on wether we are using the effective potential per unit co-moving or per unit physical volume.)

Expanding $\mathcal{V}(\lambda)$ for small λ and in $D = 5$, we have

$$\mathcal{V}(\lambda) = \mathcal{I}\lambda^2 + O\left(\lambda^4 \ln \lambda\right), \tag{4.31}$$

where

$$\mathcal{I} = -\frac{1}{2} \int_0^\infty d\rho\, \rho^5 \frac{K_1(\rho)}{I_1(\rho)} \approx -4.2. \tag{4.32}$$

Hence,

$$V_{eff}(\lambda) = \frac{\ell^{-4}}{(4\pi)^2} \left[\alpha + \beta\lambda^4 + \mathcal{I}\lambda^6 + \cdots \right]. \tag{4.33}$$

The second equality in (3.10) determines $\beta \approx -(3/2)\mathcal{I}\lambda_{obs}^2$, and then the first of the renormalization conditions gives the aditive constant α. With this, the physical mass of the radion from the point of view of the negative tension brane is given by

$$m_\phi^{2\,(-)} \approx \lambda_{obs}^2 \left(\frac{\mathcal{I}}{\pi M^3 l^5} \right). \tag{4.34}$$
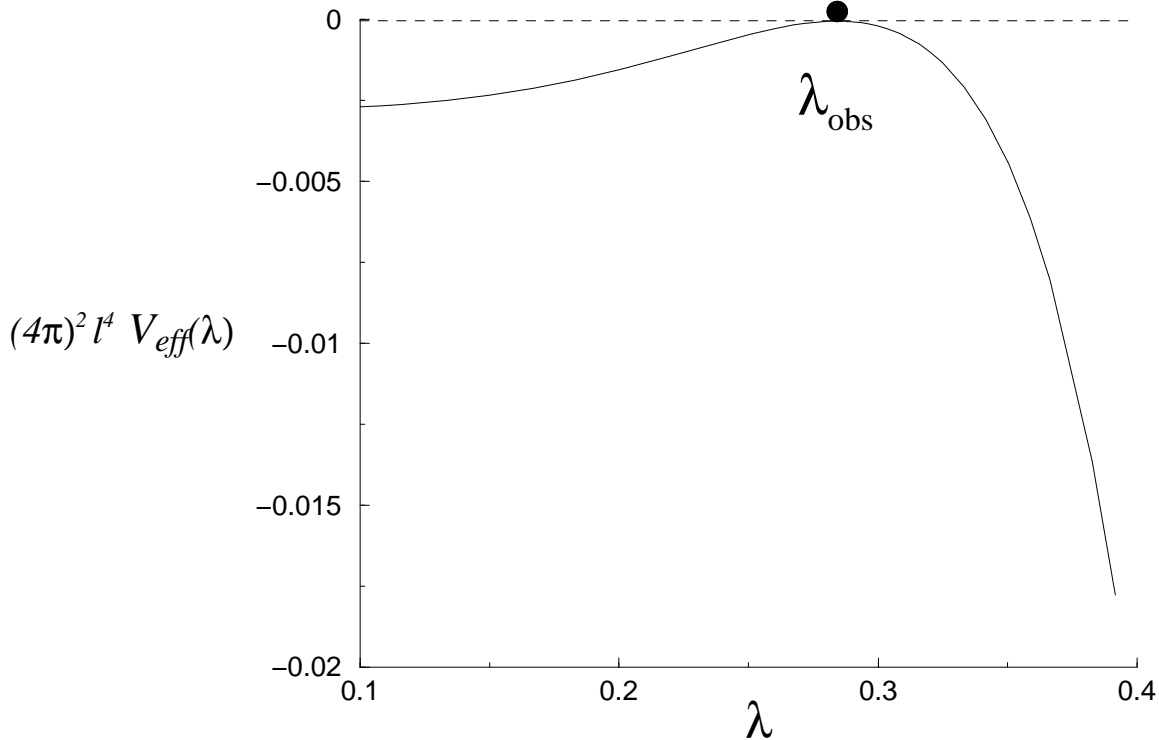
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FIG. 5. Graviton contribution to the radion effective potential.

For the case of the graviton considered here, this mass squared is negative (this is shown here for small λ_{obs}, but it is in fact true for all λ_{obs}), and therefore the graviton cannot by itself stabilize the radion.

As we have seen in Section II, in theories where additional fields can propagate in the bulk, some of them may give positive contributions to the mass squared. In fact, for a small λ_{obs}, the contribution of conformally invariant fermions $m_\phi^{2\,(-)} \sim \lambda_{obs}\,(TeV)^2$, studied in Section II, is much more important than that of gravitons, which scales as λ_{obs}^2.

V. CONCLUSIONS AND DISCUSSION

We have shown that in brane-world scenarios with a warped extra dimension, it is in principle possible to stabilize the radion ϕ through the Casimir force induced by bulk fields. In particular, conformally invariant fields induce an effective potential of the form (3.9) as measured from the positive tension brane. From the point of view of the negative tension brane, this corresponds to an energy density per unit physical volume of the order

$$V_{eff}^- \sim m_{pl}^4 \left[\frac{A\lambda^4}{(1-\lambda)^4} + \alpha + \beta\lambda^4 \right],$$

where A is a calculable number (of order 10^{-3} per degree of freedom), and $\lambda \sim \phi/(M^3\ell)^{1/2}$ is the dimensionless radion. Here M is the higher-dimensional Planck mass, and ℓ is the

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AdS radius, which are both assumed to be of the same order, whereas m_{pl} is the lower-dimensional Planck mass. In the absence of any fine-tuning, the potential will have an extremum at $\lambda \sim 1$, where the radion may be stabilized (at a mass of order m_{pl}). However, this stabilization scenario without fine-tuning would not explain the hierarchy between m_{pl} and the TeV.

A hierarchy can be generated by adjusting β according to (3.11), with $\lambda_{obs} \sim (TeV/m_{pl}) \sim 10^{-16}$ (of course one must also adjust α in order to have vanishing four-dimensional cosmological constant). But with these adjustement, the mass of the radion would be very small, of order

$$m_\phi^{2\,(-)} \sim \lambda_{obs}\ M^{-3}\ell^{-5} \sim \lambda_{obs}(TeV)^2. \tag{5.1}$$

Therefore, in order to make the model compatible with observations, an alternative mechanism must be invoked in order to stabilize the radion, giving it a mass of order TeV.

Goldberger and Wise [6,7], for instance, introduced a field v with suitable classical potential terms in the bulk and on the branes. In this model, the potential terms on the branes are chosen so that the v.e.v. of the field in the positive tension brane v_+ is different from the v.e.v. on the negative tension brane v_-. Thus, there is a competition between the potential energy of the scalar field in the bulk and the gradient which is necessary to go from v_+ to v_-. The radion sits at the value where the sum of gradient and potential energies is minimized. This mechanism is perhaps somewhat *ad hoc*, but it has the virtue that a large hierarchy and an acceptable radion mass can be achieved without much fine tuning. It is reassuring that in this case the Casimir contributions, given by (5.1), would be very small and would not spoil the model.

We have also calculated the graviton contribution to the radion effective potential. Since gravitons are not conformally invariant, the calculation is considerably more involved, and a suitable method has been developed for this purpose. The result is that gravitons contribute a negative term to the radion mass squared, but this term is even smaller than (5.1), by an extra power of λ_{obs}.

In an interesting recent paper [12], Fabinger and Horava have considered the Casimir force in a brane-world scenario similar to the one discussed in this paper, where the internal space is topologically S^1/Z_2. In their case, however, the gravitational field of the branes is ignored and the extra dimension is not warped. As a result, their effective potential is monotonic and stabilization does not occur (at least in the regime where the one loop calculation is reliable, just like in the original Kaluza-Klein compactification on a circle [32]). The question of gravitational backreaction of the Casimir energy onto the background geometry is also discussed in [12]. Again, since the gravitational field of the branes is not considered, they do not find static solutions. This is in contrast with our case, where static solutions can be found by suitable adjustment of the brane tensions.

Finally, it should be pointed out that the treatment of backreaction (here and in [12]) applies to conformally invariant fields but not to gravitons. Gravitons are similar to minimally coupled scalar fields, for which it is well known that the Casimir energy density diverges near the boundaries [13]. Therefore, a physical cut-off related to the brane effective width seems to be needed so that the energy density remains finite everywhere. Presumably, our conclusions will be unchanged provided that this cut-off length is small compared with the interbrane separation, but further investigation of this issue would be interesting.

VI. ACKNOWLEDGEMENTS

APPENDIX A: EVALUATION OF THE GENERALIZED ZETA FUNCTION

In this Appendix, we evaluate the generalized zeta function (4.26) which appears in the expression of the effective potential (4.25). We shall closely follow the method of Ref. [29,28].

The asymptotic behavior of the function F in the integrand of (4.26) is given by

$$
F(t) \to
\begin{cases}
\dfrac{i}{\pi\sqrt{\lambda}t} e^{-i(1-\lambda)t}\left(1+O(t^{-1})\right), & (it \to \infty), \\[2mm]
\dfrac{-i}{\pi\sqrt{\lambda}t} e^{i(1-\lambda)t}\left(1+O(t^{-1})\right), & (it \to -\infty).
\end{cases}
\tag{A1}
$$

Hence we rewite the zeta function as

$$
\hat\zeta(s) = \frac{s}{2\pi i}\sum_{\sigma=\pm}\left\{\int_{\mathcal{C}_\sigma} dt\, t^{-1-s}\ln\left[-\sigma\pi\sqrt{\lambda}te^{\sigma i(1-\lambda)t}F(t)\right] - \int_{\mathcal{C}_\sigma} dt\, t^{-1-s}\ln\left[-\sigma\pi\sqrt{\lambda}te^{\sigma i(1-\lambda)t}\right]\right\}. \tag{A2}
$$

Here, the contours $\mathcal{C}_\pm$ are the upper and lower halves of the contour $\mathcal{C}$ in Fig. 3.

The contribution from the circle at infinity in the first term vanishes, so this part of the contour can be dropped. After that, the domain of convergence of the expression is extended to $\Re s > -1$. The second term can be explicitly evaluated for large s as

$$
\begin{aligned}
&\int_{\mathcal{C}_\pm} dt\, t^{-1-s}\ln\left[\mp\pi\sqrt{\lambda}te^{\pm(1-\lambda)t}\right] \\
&= \mp\int_\varepsilon^\infty dt\, t^{-1-s}\ln\left[\mp\pi\sqrt{\lambda}te^{\pm(1-\lambda)t}\right] \\
&= \mp\left\{\left(\frac{\ln[\mp i\pi\sqrt{\lambda}]}{s}+\frac{1}{s^2}\right)\varepsilon^{-s}+\frac{1}{s}\varepsilon^{-s}\ln\varepsilon \mp\frac{i(1-\lambda)}{1-s}\varepsilon^{1-s}\right\}.
\end{aligned}
\tag{A3}
$$

Thus, the whole expression can be analytically continued to the strip given by $-1 < \Re s < 0$. Taking the $\varepsilon \to 0$ limit after this analytic continuation, we obtain

$$
\begin{aligned}
\hat\zeta(s) &= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_0^\infty d\rho\, \rho^{-1-s}\ln\left[\pi\sqrt{\lambda}\rho e^{-(1-\lambda)\rho}F(i\rho)\right] \\
&= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_0^\infty d\rho\, \rho^{-1-s}\ln\left[2\sqrt{\lambda}\rho e^{-(1-\lambda)\rho}\left\{I_{\eta-1}(\rho)K_{\eta-1}(\lambda\rho)-I_{\eta-1}(\lambda\rho)K_{\eta-1}(\rho)\right\}\right].
\end{aligned}
\tag{A4}
$$

The asymptotic expansion of the modified Bessel functions is given by

$$
\begin{aligned}
I_\nu(t) &\sim \frac{e^t}{\sqrt{2\pi t}}C_\nu(-t)+O(e^{-t}), \\
K_\nu(t) &\sim \sqrt{\frac{\pi}{2t}}e^{-t}C_\nu(t),
\end{aligned}
\tag{A5}
$$

with

$$C_\nu(t) = \sum_{r=0}^{\infty} \frac{\Gamma\left(\nu + r + \frac{1}{2}\right)}{r!\,\Gamma\left(\nu - r + \frac{1}{2}\right)}(2t)^{-r}. \tag{A6}$$

We define coefficients β_r by

$$\ln\left[C_{\eta-1}(\rho)\right] = \sum_{r=1}^{\infty} \frac{\beta_r}{\rho^r}. \tag{A7}$$

Now, we rewrite the expression for $\hat{\zeta}(s)$ as

$$\hat{\zeta}(s) = \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\left\{\lambda^s \int_0^\infty dt\, t^{-1-s}\ln\left[\sqrt{2t/\pi}\,e^t K_{\eta-1}(t)\right]\right.$$
$$+ \int_0^\infty d\rho\,\rho^{-1-s}\ln\left[\sqrt{2\pi\rho}\,e^{-\rho} I_{\eta-1}(\rho)\right]$$
$$\left. + \int_0^\infty d\rho\,\rho^{-1-s}\ln\left[1 - \frac{I_{\eta-1}(\lambda\rho)}{K_{\eta-1}(\lambda\rho)}\frac{K_{\eta-1}(\rho)}{I_{\eta-1}(\rho)}\right]\right\}. \tag{A8}$$

The first integral in the square brackets can be evaluated in the following way.

Defining the function

$$R(t) = \sum_{r=1}^{D-2} \frac{\beta_r}{t^r} + \frac{\beta_{D-1}}{t^{D-1}}e^{-\left|\frac{1}{t}\right|}, \tag{A9}$$

we can rewrite that integral as

$$\int_0^\infty dt\, t^{-1-s}\ln\left[\sqrt{2t/\pi}\,e^t K_{\eta-1}(t)\right] = \int_0^{t_0} dt\, t^{-1-s}\ln\left[\sqrt{2t/\pi}\,e^t K_{\eta-1}(t)\right]$$
$$+ \int_{t_0}^\infty dt\, t^{-1-s}\left(\ln\left[\sqrt{2t/\pi}\,e^t K_{\eta-1}(t)\right] - R(t)\right)$$
$$+ \sum_{r=1}^{D-2} \frac{\beta_r}{s+r}t_0^{-s-r} + \beta_{D-1}\int_{t_0}^\infty dt\, t^{-s-D}e^{-\left|\frac{1}{t}\right|}. \tag{A10}$$

Except for poles, this expression is now analytic in the strip $-(D-2) < \Re s < 0$. The integral in the last term is given by

$$\int_{t_0}^\infty dt\, t^{-s-D}e^{-\frac{1}{t}} = \int_0^{\frac{1}{t_0}} dx\, x^{s+(D-1)-1}e^{-x}$$
$$= \frac{1}{s+(D-1)}\left\{t_0^{-s-(D-1)}e^{-\frac{1}{t_0}} + \int_0^{\frac{1}{t_0}} dx\, x^{s+(D-1)}e^{-x}\right\}. \tag{A11}$$

After substitution of this expression in (A10), we can perform analytic continuation to $-(D-2) > \Re s > -D$. Then, taking the $t_0 \to 0$ limit, the right hand side of Eq. (A10) becomes

$$\int_0^\infty dt\, t^{-1-s}\left(\ln\left[\sqrt{2t/\pi}\,e^t K_{\eta-1}(t)\right] - R(t)\right) + \frac{\beta_{D-1}\Gamma(s+D)}{s+(D-1)}. \tag{A12}$$

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Using $\Gamma(1+x) = 1 - \gamma x + O(x^2)$, we find

$$\int_0^\infty dt\, t^{-1-s} \ln\left[\sqrt{2t/\pi}\, e^t K_{\eta-1}(t)\right] = \frac{\beta_{D-1}}{s+(D-1)} + \mathcal{I}_K + O\left(s+(D-1)\right), \qquad (A13)$$

where

$$\mathcal{I}_K = -\gamma\beta_{D-1} + \int_0^\infty dt\, t^{D-2}\left(\ln\left[\sqrt{2t/\pi}\, e^t K_{\eta-1}(t)\right] - R(t)\right). \qquad (A14)$$

The second term in the square brackets in (A8) can be evaluated in a similar way as

$$\int_0^\infty d\rho\, \rho^{-1-s} \ln\left[\sqrt{2\pi\rho}\, e^{-\rho} I_{\eta-1}(\rho)\right] = \frac{\beta_{D-1}}{s+(D-1)} + \mathcal{I}_I + O\left(s+(D-1)\right), \qquad (A15)$$

where

$$\mathcal{I}_I = -\gamma\beta_{D-1} + \int_0^\infty d\rho\, \rho^{D-2}\left(\ln\left[\sqrt{2\pi\rho}\, e^{-\rho} I_{\eta-1}(\rho)\right] - R(-\rho)\right). \qquad (A16)$$

Substituting the above results into (A8), we obtain

$$\hat{\zeta}(1-D) = -(-1)^\eta \eta\beta_{D-1}(1+\lambda^{1-D}) = \begin{cases} \dfrac{1}{16}(1+\lambda^{-2}), & (D=3), \\[2mm] \dfrac{27}{64}(1+\lambda^{-4}), & (D=5), \end{cases} \qquad (A17)$$

and

$$\hat{\zeta}'(1-D) = -(-1)^\eta \eta\left[\mathcal{I}_I + \lambda^{1-D}\mathcal{I}_K + \beta_{D-1}\left(\frac{1+\lambda^{1-D}}{1-D} + \lambda^{1-D}\ln\lambda\right) + \mathcal{V}(\lambda)\right], \qquad (A18)$$

where $\mathcal{V}(\lambda)$ is given in (4.28). Then, substituting these results into (4.25), we obtain (4.27).

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